Exploring Pi, Infinities, and Normality

A Study on Decimal Expansion and Set Cardinality

Matthew West

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Abstract

This paper explores the properties of the transcendental number π, specifically with regard to the unproven conjecture of its normality. We investigate the logical constraints of infinite decimal expansions, the impossibility of terminal digits in infinite sequences (the "After-Infinity" Paradox), and the counterintuitive nature of countable versus uncountable infinities as defined by Georg Cantor.

1 The Normality of π

It is a well-established fact that π is an irrational and transcendental number. Its decimal expansion is infinite and non-periodic. However, a common misconception is that infinity implies the inclusion of all possible finite strings (e.g., a birthdate or a Social Security Number).

As the author observes, while π never repeats a pattern, we do not yet know if it is a **normal number**. A number is normal if every sequence of k digits appears with a frequency of 10^{-k}. Without a proof of normality, it is theoretically possible for π to be infinite yet "boring"—for example, it could eventually stop using the digit "7" entirely after a certain point.

2 The Paradox of the Terminal Digit

A significant portion of this research addresses the logical constraints of infinite strings: the fact that nothing can exist "after" an infinite sequence.

Consider the following theoretical string:

$$0.999\ldots0 \tag{1}$$

In standard real number analysis, this value is non-existent. Because every digit in a decimal expansion must correspond to a natural number position $n \in \mathbb{N}$, and there is no "final" natural number, there can be no digit placed at the "end" of an infinite sequence.

This has a profound implication for normality: while a birthday can appear an infinite number of times within π, it can never repeat *consecutively* forever, as that would convert the number into a repeating rational fraction.

3 Comparing Sizes of Infinity

Finally, we address the cardinality of sets. It is a common intuition that the set of all natural numbers ($\mathbb{N}$) must be larger than the set of even numbers ($2\mathbb{N}$). However, Cantor proved that they are the same size, denoted as $\aleph_0$ (Aleph-null).

3.1 The Bijection Proof

Two sets are the same size if a one-to-one correspondence (bijection) exists between them:

$$\begin{gathered} 1 \leftrightarrow 2 \\ 2 \leftrightarrow 4 \\ 3 \leftrightarrow 6 \\ \cdots \\ n \leftrightarrow 2n \end{gathered}$$

Since every "whole" number has a unique "even" partner, the sets are equal in size.

3.2 Uncountable Sets and Power Set Cardinality

The transition from countable to uncountable infinity is best understood through Cantor's Theorem. Cantor proved that for any set S, the power set $P(S)$ (the set of all possible subsets) has a strictly larger cardinality than S.

To bridge the gap between the power set of the natural numbers and the real numbers, we consider that any real number can be represented as an infinite binary sequence. These sequences correspond directly to the elements of $P(\mathbb{N})$. Therefore:

$$|\mathbb{R}| = |P(\mathbb{N})| = 2^{\aleph_0} \tag{2}$$

Using the **Diagonal Argument**, Cantor showed that no bijection can exist between $\mathbb{N}$ and $\mathbb{R}$, proving that:

$$2^{\aleph_0} > \aleph_0 \tag{3}$$

4 Conclusion

The study of π and infinity reveals a universe that is both chaotic and strictly ordered. While it may feel "sad" that we cannot place a zero at the end of an infinite string of nines, this mathematical boundary is exactly what allows for the existence of transcendental numbers. Infinity is not just "a very big number"; it is a collection of different magnitudes that redefine our understanding of logic.